The Fundamental Theorem of Utility Maximization and Numéraire Portfolio*

Tahir Choulli†, Jun Deng and Junfeng Ma

Mathematical and Statistical Sciences Dept.
University of Alberta, Edmonton, Alberta

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Abstract

The fundamental theorem of utility maximization (called FTUM hereafter) says that the utility maximization admits solution if and only if there exists an equivalent martingale measure. This theorem is true for discrete market models (where the number of scenarios is finite), and remains valid for general discrete-time market models when the utility is smooth enough. However, this theorem –in this current formulation– fails in continuous-time framework even with nice utility function, where there might exist arbitrage opportunities and optimal portfolio. This paper addresses the question how far we can weaken the non-arbitrage condition as well as the utility maximization problem to preserve their complete and strong relationship described by the FTUM. As application of our version of the FTUM, we establish equivalence between the No-Unbounded-Profit-with-Bounded-Risk condition, the existence of numéraire portfolio, and the existence of solution to the utility maximization under equivalent probability measure. The latter fact can be interpreted as a sort of weak form of market's viability, while this equivalence is established with a much less technical approach. Furthermore, the obtained equivalent probability can be chosen as close to the real-world probability measure as we want (but might not be equal).

1 Introduction

Since its birth in Bachelier's thesis, mathematical finance has three pillars with immense success among many topics. These pillars are utility maximization, non-arbitrage, and equivalent martingale measures (called UM, NA, and EMM respectively hereafter). The last two pillars have been linked to each other for particular models by Kreps in [34], Harrison–Pliska in [23], Dalang–Morton–Willinger in [11]. This equivalence between NA and existence of EMM has been termed in the literature as the fundamental theorem of asset pricing (FTAP hereafter). For the most general framework, Delbaen and Schachermayer strengthened the non-arbitrage condition (by considering No-Free-Lunch-with-Vanishing-Risk), and weakened the EMM (by considering σ-martingale measures), in order to establish the very general version of the FTAP with the same original spirit in their seminal works [13] and [14]. The FTAP had been extended very successfully in many directions such as markets with proportional transaction costs, where the works of Kabanov and Jouini/Kallal (see [25] and [21]) play a foundational rôle in

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†corresponding author, Email: tchoulli@ualberta.ca

the literature.

The equivalence between the UM and NA (or UM and EMM) was proved in discrete markets (when the sample space has finite number of scenarios) and it goes back to Arrow and Debreu (see[1] and [10]). Inspired by the FTAP terminology, Kallsen baptized this equivalence between EMM and UM (see [26]) as the fundamental theorem of utility maximization (FTUM hereafter). This theorem was extended to the general discrete-time framework with smooth enough utility function. We refer the reader to [26] and [36] for this extension. In the latter paper, the authors discussed weaker assumptions on the utility that imply the existence of the optimal solution. The utility maximization problem (called also Merton's problem) had been intensively investigated under the existence of EMM (the strongest form of NA) which allows authors to use the two rich machineries of martingale theory and convex duality. These works can be traced back to [32], [18], [15] and [27], and the references therein to cite few. The main results in this literature focus on finding assumptions on the utility function for which versions of duality can hold, and/or solutions to both the primal problem and its dual problem will exist. Thus, the question of how the existence of optimal portfolio is connected to NA (weak or strong form) has been forgotten for the continuous-time context. Recently, Frittelli addressed this relationship in [19] from a different angle. The author defined a market's non-arbitrage using classes of agent's preferences and connected his new non-arbitrage concepts to the existing ones. He also proved the existence of an absolutely continuous separating measure with nice integrability feature, whenever the optimal value function of the utility maximization is finite. On one hand, this last statement does not imply non-arbitrage because of the absence of equivalence between measures within the statement. On the other hand, [19] as whole work, can not be applied to model of [35] and Example of Section 3.

Herein, we are interested in deriving the exact version of the FTUM –à la Delbaen and Schachermayer– by weakening the non-arbitrage condition and/or the utility maximization problem. The obtained version of the FTUM constitutes our first main result, while its application to numéarire portfolio and market's weak viability represents our second main result. The numéraire portfolio is another interesting problem that took large part of the literature since early nineties by the works of Artzner, Becherer, Bühlmann, Christensen, Karatzas, Kardaras, Korn, Larsen, Long, Platen, Schäl, and Sass (see [2], [5], [31], [33], [9], [39], and the references therein). Herein, our main contribution to this extensive literature lies in the approach (which is less technical comparing with [31] for instance) and more importantly in establishing close ties between numéraire portfolio and the existence of solution of utility maximization for a larger class of utilities (not only log) without any assumption. Precisely, by using equivalent change of probability, we elaborate the equivalence described above. This change of probability might be economically interpreted as a weak form of market's viability.

This paper is organized as follows. In the next section, we will describe the mathematical model and present some preliminaries that will be useful through out the paper. The third section contains the new version of the Fundamental Theorem of Utility Maximization. This version is established for utilities with effective domain $(0, +\infty)$ and $\mathbb{R}$, and for exponential utility in three separate subsections. The last section is devoted to our second main contribution which lies in numéraire portfolio and market's weak viability.

2 Preliminaries

We start by a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$, where the filtration satisfies the usual condition of right continuity and completeness. On this stochastic base, we consider a d-dimensional semi-martingale $(S_t)_{0\leq t\leq T}$, that represents the discounted price of d-risky assets. The space of mar-

tingales will be denoted by $\mathcal{M}(P)$, and the set of predictable processes that are S-integrable will be denoted by $L(S)$.
As usual, $\mathcal{A}^+(Q)$ denotes the set of increasing, right-continuous, adapted and Q-integrable processes.

If $\mathcal{C}$ is a class of processes, we denote by $\mathcal{C}_0$ the set of processes X that belong to $\mathcal{C}$ with $X_0 = 0$ and by $\mathcal{C}_{loc}$ the set of processes X such that there exists a sequence of stopping times, $(T_n)_{n\geq 1}$, increasing stationarily to T (i.e., $P(T_n = T) \to 1$ as $n \to \infty$) and the stopped process X^{T_n} belongs to $\mathcal{C}$. We put $\mathcal{C}_{0,loc} = \mathcal{C}_0 \cap \mathcal{C}_{loc}$. Below, we recall a lemma from [20] (see Lemma 1.35) that addresses the localization stability.

Lemma 2.1. *Let $\mathcal{C}$ and $\mathcal{C}'$ be two classes of processes that are stable under stopping. Then*
(a) $\mathcal{C}_{loc}$ is stable under stopping and $(\mathcal{C}_{loc})_{loc} = \mathcal{C}_{loc}$.
(b) $(\mathcal{C} \cap \mathcal{C}')_{loc} = \mathcal{C}_{loc} \cap \mathcal{C}'_{loc}$.

Below we define the admissibility for strategies that will be used through out the paper.

Definitions 2.2. *Let $H = (H_t)_{0\leq t\leq T}$ be a predictable process.*
(i) For any positive constant $\alpha > 0$, H is called α-admissible if H is S-integrable and $(H \cdot S)_t \geq -\alpha,\ P-a.s$ for any $t \in [0,T]$.
(ii) We say H is admissible if there exists a positive constant α such that H is α-admissible.

For any $x > 0$, we define the set of wealth processes obtained from admissible strategies with initial capital x by

$$\begin{aligned}
&\mathcal{X}(x) := \{X \geq 0 \mid \exists\, \theta \in L(S),\ X = x + \theta \cdot S\ \},\ \mathcal{X}_+(x) := \{X \in \mathcal{X}(x) \mid X > 0\ \},\\
&\mathcal{K}(x) := \left\{Y \in L^0(P) \mid \exists\, X \in \mathcal{X}(x),\ \ Y = X_T - x\right\}.
\end{aligned} \tag{2.1}$$

Definitions 2.3. *S satisfies no-arbitrage (called hereafter NA) if for any admissible strategy H such that $(H \cdot S)_T \geq 0$, implies $(H \cdot S)_T = 0$, $P-a.s.$*

Definitions 2.4. *S satisfies the no-unbounded-profit-with-bounded-risk (called hereafter NUPBR) condition if the set $\mathcal{K}(1)$ is bounded in probability.*

Definitions 2.5. *A process X is called a σ-martingale, if it is a right continuous with left limit (called RCLL hereafter) semimartingale for which there exists a predictable process ϕ such that $0 < \phi \leq 1$ and $(\phi \cdot S)$ is a local martingale.*

The definition of σ-martingale goes back to Chou [6] (see also [16]). It results naturally when we integrate –in the semimartingale sense– an unbounded and predictable process with respect to a local martingale. The difference between σ-martingale and local martingale is discussed by Ansel and Stricker in [4].

Definitions 2.6. *A σ-martingale density for S is any positive local martingale, Z, such that ZS is a σ-martingale. The set of σ-martingale densities for S will be denoted by*

$$\mathcal{Z}_{loc}(S) := \left\{Z \in \mathcal{M}_{loc}(P)\ \middle|\ Z > 0,\quad ZS \textit{ is a } \sigma\textit{-martingale}\ \right\} \tag{2.2}$$

Remark: For any $Z \in \mathcal{Z}_{loc}(S)$ and any $\theta \in L(S)$ such that $\theta \cdot S \geq a$ ($a \in \mathbb{R}$), the process $Z(\theta \cdot S)$ is a local martingale. This follows immediately from Proposition 3.3 and Corollary 3.5 of [4].
The exact connection between $\mathcal{Z}_{loc}(S) \neq \emptyset$ and NUPBR had been established in full generality by Takaoka (see [40] for details). This result can be traced back to Choulli and Stricker for the case of continuous processes (see [7]), and to Kardaras for the case one-dimensional general semimartingale framework (see [30]). Below, we state this important result in its full generality.

Theorem 2.7. *Let S be a semimartingale. Then, S satisfies the NUPBR condition if and only if $\mathcal{Z}_{loc}(S) \neq \emptyset$.*

Beside non-arbitrage concepts, our second main ingredient –that we will deal with through out the paper– is the agent's preference (utility function). Below, we provide its definition and the corresponding admissible set of strategies.

Definitions 2.8. *A utility function is a function that we denote by U, and for which there exists $u_0 \in [-\infty, 0]$ such that $(u_0, +\infty)$ contains the effective domain of U, and U is continuously differentiable, strictly increasing and strictly concave on this domain.*

For a utility function U, a semimartingale X and a probability Q, the set of admissible portfolios will be denoted by

$$\mathcal{A}_{adm}(a, U, X, Q) := \Big\{ H \mid \; H \in L(X), \; H \cdot X \geq -a \;\; \& \;\; E^Q\Big[U^-(a + (H \cdot S)_T)\Big] < +\infty \Big\}. \qquad (2.3)$$

When $Q = P$ and U is fixed, we simply denote $\mathcal{A}_{adm}(a, S)$.
The following lemma is a variant of Kolmos' argument and is borrowed from [13].

Lemma 2.9. *(see Lemma A1.1 in [13])*
Let (f_n) be a sequence of $[0, +\infty[$ valued measurable functions on a probability space $(\Omega, \mathcal{F}, P)$. There is a sequence $g_n \in conv(f_l, \; l \geq n)$ such that g_n converges almost surely to a $[0, +\infty]$ valued function g, and the following properties hold:
(1) If $conv(f_n, n \geq 1)$ is bounded in L^0, then g is finite almost surely,
(2) If there are $c > 0$ and $\delta > 0$ such that for all n

$$P(f_n > c) > \delta,$$

then $P(g > 0) > 0$.

The convergence of the sequence in this lemma is termed in the literature by Fatou's convergence. The dynamic version of this convergence (given below) is borrowed from [17] and has been frequently used in the literature.

Definitions 2.10. *Let $\mathcal{J}$ be a dense subset of $\mathbb{R}_+$. A sequence of processes (X_n) is called Fatou convergent on $\mathcal{J}$ to a process X if (X_n) is uniformly bounded from below, and if for any $t \geq 0$ we have*

$$X_t = \limsup_{s \downarrow t, s \in \mathcal{J}} \limsup_{n \to \infty} X^n_s = \liminf_{s \downarrow t, s \in \mathcal{J}} \liminf_{n \to \infty} X^n_s.$$

If $\mathcal{J} = \mathbb{R}_+$, the sequence (X_n) is called simply Fatou convergent.

Therefore, the dynamic version of Lemma 2.9 can be found, again, in [17] and is recalled below.

Lemma 2.11. *(1) Let (X_n) be a sequence of supermartingales which are uniformly bounded from below such that $X_n \geq 0$. Let $\mathcal{J}$ be a dense countable subset of $\mathbb{R}_+$. Then there is a sequence $Y_n \in conv(X^n, X^{n+1}, \cdots)$ and a supermartingale Y such that $Y_0 \leq 0$ and (Y_n) is Fatou convergent on $\mathcal{J}$ to Y.*
(2) Let $(A_n)_{n \geq 1}$ be a sequence of increasing processes such that $A^n_0 = 0$. There is a sequence $B^n \in conv(A^n, A^{n+1}, \cdots)$ and an increasing process B with values in $\overline{R}_+$ such that (B^n) is Fatou convergent to B. If there are $T > 0, a > 0$ and $\delta > 0$ such that $P(A^n_T > \alpha) > \delta$ for all $n \geq 1$, then $P(B_T > 0) > 0$.

The importance of this lemma lies mainely in the optional decomposition of Kramkov (see [28]), that we will use in our proof. In fact, we will use its weakest form that was elaborated by Stricker and Yan in Theorem 2.1 of [38], where the authors used the set $\mathcal{Z}_{loc}(S)$ instead of the set of σ-martingale measures. As a direct consequence of Lemma 2.11 and Theorem 2.1 of [38], we obtain the following

Corollary 2.12. *Suppose that $\mathcal{Z}_{loc}(S) \neq \emptyset$. Let $(\theta_n)_{n\geq 1}$ such that $\theta_n \in L(S)$ and $\theta_n \cdot S \geq -1$. Then there exist $\widehat{\theta} \in L(S)$ and a nondecreasing process C such that $\widehat{\theta} \cdot S \geq -1$, $C_0 = 0$, and*

$$1 + \theta_n \cdot S \quad \textit{Fatou convergeges to} \quad 1 + \widehat{\theta} \cdot S - C. \tag{2.4}$$

3 The Fundamental Theorem of Utility Maximization

This section discusses one of the main contributions of the paper. Precisely, we describe the exact relationship between existence of solution to *the localized* utility maximization problem and the NUPBR concept. By the localized utility maximization problem, we mean the sequence of utility maximization problems obtained when stopping S with a sequence of stopping times that increases stationarily to T. The intuitive idea behind this connection lies in the simple remark that the NUPBR concept is stable under the localization procedure. This stability is stated below.

Proposition 3.1. *Suppose that there exists a sequence of stopping times $(\tau_n)_{n\geq 0}$ increasing stationarily to T, such that for each $n \geq 0, S^{\tau_n}$ satisfies NUPBR. Then, S satisfies NUPBR.*

Proof. Let $c > 0$, and $n \geq 1$. Then, if we put

$$\mathcal{K}(1, S) := \{\theta \in L(S) \ \ | \ \ \theta \cdot S \geq -1\},$$

then $\mathcal{K}(1, S) \subset \mathcal{K}(1, S^{\tau})$ for any stopping time τ. Thus we derive

$$\sup_{\theta\in\mathcal{K}(1,S)} P\Big((\theta \cdot S)_T \geq c\Big) \quad \leq \quad \sup_{\theta\in\mathcal{K}(1,S^{\tau_n})} P\Big((\theta \cdot S)_{\tau_n} \geq c\Big) + P(T > \tau_n).$$

Then, by taking the letting c goes to infinity, we deduce that

$$\lim_{c\longrightarrow+\infty} \sup_{\theta\in\mathcal{K}(1,S)} P\Big((\theta \cdot S)_T \geq c\Big) \leq P(T > \tau_n),$$

for any $n \geq 1$. This obviously implies the boundedness in probability of $\{(\theta\cdot S)_T \ \Big| \ \theta \in L(S) \ \ \theta\cdot S \geq -1\}$. This ends the proof of the proposition. ☐

Remark: It is very clear from the previous proposition that NUPBR is a dynamic/local property for S, while the NA property is defined in static manner. More importantly, we can not substitute the NUPBR by NA in Proposition 3.1. In other words, in general, we can find a sequence of stopping times $(\tau_n)_{\geq 1}$ increasing stationarily to T, and S such that S^{τ_n} satisfies NA, while NA fails globally (for S). This is one of the main idea behind Delbaen and Schachermayer (1995). For the sake of completeness, below we provide a counterexample of S satisfying No-Free-Lunch-with-Vanishing-Risk (see [12] and [13] for the definition of this condition that we call NFLVR hereafter) locally, but the NFLVR fails globally.

Example: Consider the following market model where there is one stock on the finite time horizon $[0, 1]$, with $S_0 = 1$ and S satisfying the stochastic differential equation $dS_t = (1/S_t)dt + d\beta_t$. Here β

is a standard, one dimensional Brownian motion, so S is the three-dimensional Bessel process. This example was considered in quite a number of papers starting with [12], [3], and [31]. Put,

$$dX_t := dS_t/S_t = (1/S_t^2)dt + (1/S_t)d\beta_t.$$

The numéraire portfolio exists and is given by $1/S = \mathcal{E}(-\frac{1}{S}\cdot\beta)$ which is a local martingale. Then, it is easy to calculate

$$\log(\mathcal{E}(X)_T) = \int_0^T \frac{1}{S_u}d\beta_u + \frac{1}{2}\int_0^T \frac{1}{S_u^2}du, \quad \text{and } E\int_0^T \frac{1}{S_u^2}du \le 8 + T < +\infty.$$

This proves that $\frac{1}{S}\cdot\beta$ is a square integrable martingale and $\log(\mathcal{E}(X)_T)$ is an integrable random variable. Hence, a combination of this with the supermartingale of $\frac{1+\theta\cdot S}{\mathcal{E}(X)}$ for any $\theta \in L(S)$ such that $1+\theta\cdot S > 0$, we obtain

$$E\left[\log(1+\theta\cdot S_1)\right] - E\left[\log(\mathcal{E}(X)_1)\right] = E\left[\log\left(\tfrac{1+\theta\cdot S_1}{\mathcal{E}(X)_1}\right)\right] \le \log\left(E\left[\tfrac{1+\theta\cdot S_1}{\mathcal{E}(X)_1}\right]\right) \le 0.$$

Then, we deduce that both the optimal portfolio and arbitrage opportunities exist (see [31]). Hence, this example provides evidence that the existence of solution to the utility maximization does not guarantee the existence of equivalent martingale measure. Therefore, the Fundamental Theorem of Utility Maximization –in its original form as stated in the abstract– is violated for this continuous-time model even when the utility is so nice.

3.1 The case of positive wealth

In this subsection, we focus on the case where the utility function U satisfies

$$\mathrm{dom}(U) = (0,+\infty), \quad U'(0) = +\infty, \quad U'(\infty) = 0, \ \ \& \quad \limsup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1, \tag{3.5}$$

where $\mathrm{dom}(U)$ denotes the effective domain of U (i.e. the set of $x \in \mathbb{R}$ such that $-\infty < U(x)$). Below, we state our main result of this subsection.

Theorem 3.2. *Let U be a utility function such that $dom(U) = (0,+\infty)$ and (3.5) holds. Suppose that there exists a sequence of increasing stopping times $(T_n)_{n\ge1}$ that increases stationarily to T and $x_n > 0$ such that*

$$\sup_{\theta\in\mathcal{A}_{adm}(x_n,S^{T_n})} EU\Big(x_n + (\theta\cdot S)_{T_n}\Big) < +\infty, \qquad \forall\, n \ge 1. \tag{3.6}$$

Then, the following are equivalent:
(1) There exists a sequence of stopping times $(\tau_n)_{n\ge1}$ that increases stationarily to T such that for any $n \ge 1$ and any initial wealth $x_0 > 0$, there exists $\widehat{\theta}^{(n)} \in \mathcal{A}_{adm}(x_0, S^{\tau_n})$ such that

$$\max_{\theta\in\mathcal{A}_{adm}(x_0,S^{\tau_n})} EU\Big(x_0 + (\theta\cdot S)_{\tau_n}\Big) = EU\Big(x_0 + (\widehat{\theta}^{(n)}\cdot S)_{\tau_n}\Big) < +\infty \tag{3.7}$$

(2) S satisfies NUPBR.

Notice that the most innovative part of this theorem is the implication (1) $\Longrightarrow$ (2). It is worthwhile to notice that for this implication the assumption (3.6) is irrelevant. In our view, this theorem establishes the exact connection between the utility maximization and the Non-arbitrage condition for the general semimartingale framework. The reverse implication follows from the seminal work of Kramkov and Schachermayer (see [27]) as it will be detailed in the proof.

Proof. of Theorem 3.2: We start proving the easiest part of the theorem, that is (2) $\Longrightarrow$ (1). Thus, suppose that S satisfies NUPBR. Thanks to Theorem 2.7, we conclude the existence of a local martingale $Z > 0$ and a predictable process φ such that $0 < \varphi \leq 1$ and $Z(\varphi \cdot S) \in \mathcal{M}_{loc}(P)$. Then, for any $\theta \in L(S)$ we have $\theta \cdot S = \theta^{\varphi} \cdot S^{\varphi}$ where $\theta^{\varphi} := \theta/\varphi$ and $S^{\varphi} := \varphi \cdot S$. Thus, without loss of generality, we assume that ZS is a local martingale. Consider an increasing sequence of stopping times, $(\sigma_n)_{n\geq 1}$, that increases stationarily to T such that both Z^{σ_n} and $Z^{\sigma_n} S^{\sigma_n}$ are true martingales. Put

$$Q_n := Z_{\sigma_n} \cdot P, \quad \tau_n := T_n \wedge \sigma_n \uparrow T.$$

Then, Q_n is an equivalent martingale measure for S^{σ_n}. Since $\theta I_{[\![0,\tau_n]\!]} \in \mathcal{A}_{adm}(x_n, S^{T_n})$ whenever $\theta \in \mathcal{A}_{adm}(x_n, S^{\tau_n})$, we derive

$$\sup_{\theta \in \mathcal{A}_{adm}(x_n, S^{\tau_n})} EU\Big(x_n + (\theta \cdot S)_{\tau_n}\Big) \leq \sup_{\varphi \in \mathcal{A}_{adm}(x_n, S^{T_n})} EU\Big(x_n + (\varphi \cdot S)_{T_n}\Big) < +\infty, \qquad \forall n.$$

Therefore, a direct application of Theorems 2.1 and 2.2 of [27] implies that for any $n \geq 0$ and initial wealth $x_0 > 0$, there exists an x_0-admissible optimal strategy $\widehat{\theta}^{(n)}$ for S^{τ_n}, such that

$$\max_{\theta \in \mathcal{A}_{adm}(x_0, S^{\tau_n})} EU\Big(x_0 + (\theta \cdot S)_{\tau_n}\Big) = EU\Big(x_0 + (\widehat{\theta}^{(n)} \cdot S)_{\tau_n}\Big) < +\infty.$$

This proves assertion (1). In the remaining part of the proof we focus on proving (1) $\Longrightarrow$ (2). Suppose that assertion (1) holds, and consider $x_0 = 1 + r$ such that $r \in \mathrm{dom}(U)$ (i.e. $U(r) > -\infty$). Then,

$$\max_{\theta \in \mathcal{A}_{adm}(1+r, S^{\tau_n})} EU\Big(1 + r + (\theta \cdot S)_{\tau_n}\Big) = EU\Big(1 + r + (\widehat{\theta} \cdot S)_{\tau_n}\Big) < +\infty.$$

For the sake of simplicity, we put $\tau := \tau_n$ in what follows. Then, the NUPBR for S^{τ} is equivalent to the boundeness in probability of

$$\mathcal{K} := \{(H \cdot S)_{\tau} | H \text{ is a 1-admissible strategy for } S^{\tau}\}.$$

Suppose that $\mathcal{K}$ is not bounded in $L^0(P)$. Then, there exist a sequence of 1-admissible strategy $(\theta^m)_{m\geq 1}$, a sequence of positive real numbers, $(c_m)_{m\geq 1}$, that increases to $+\infty$, and $\alpha > 0$ such that

$$P\Big((\theta^m \cdot S)_{\tau} \geq c_m\Big) > \alpha > 0.$$

Consider a sequence of positive numbers, $(\delta_n)_{n\geq 0}$, such that

$$0 \leq \delta_m \to 0, \qquad \text{and} \qquad \delta_m c_m \to +\infty.$$

It is obvious that such sequence always exists (take $\delta_n := 1/\sqrt{c_m}$). Then, put

$$X_m := \delta_m (\theta^m \cdot S)_{\tau} \geq -\delta_m, \forall m \geq 1.$$

Hence, an application of Kolmos's argument to $(X_n + \delta_n)_{n\geq 0}$ (see Lemma 2.9), we deduce the existence of a sequence of random variables, $(g_n)_{n\geq 1}$, such that

$$0 \leq g_n := \sum_{m=n}^{N_n} \alpha_m X_m + \sum_{m=n}^{N_n} \alpha_m \delta_m \in \mathrm{conv}\Big(X_m + \delta_m,\ m \geq n\Big),$$

and g_n converges almost surely to $\widetilde{X} \geq 0$, with $P(\widetilde{X} > 0) > 0$.
Since $y_n := \sum_{m=n}^{N_n} \alpha_m \delta_m$ converges to zero, we conclude that

$$-y_n \leq \widetilde{X}_n := \sum_{m=n}^{N_n} \alpha_m \delta_m (\theta^m \cdot S)_\tau \text{ converges to } \widetilde{X} \quad P-a.s., \text{ and}$$

$$-(1+r)(1-y_n) \leq \widehat{X}_n := (1-y_n)(\widehat{\theta} \cdot S)_\tau \text{ converges to } \quad (\widehat{\theta} \cdot S)_\tau \quad P-a.s.$$

Consider the new trading strategies

$$\widetilde{\theta}^{(n)} := \sum_{m=n}^{N_n} \alpha_m \delta_m \theta_m + \Big(1 - \sum_{m=n}^{N_n} \alpha_m \delta_m\Big)\widehat{\theta} = \sum_{m=n}^{N_n} \alpha_m \delta_m \theta_m + (1-y_n)\widehat{\theta}.$$

Then, it is easy to check that $1 + r + \widetilde{\theta}^{(n)} \cdot S_\tau = 1 + r + \widetilde{X}_n + \widehat{X}_n \geq y_n r > 0$ (due mainely to $-y_n \leq \widetilde{X}_n$ and $-(1+r)(1-y_n) \leq \widehat{X}_n$). Furthermore, due to the concavity of U, we have

$$\begin{aligned}
U\Big(1 + r + (\widetilde{\theta}^{(n)} \cdot S)_\tau\Big) &= U\Big(1 + r + \widetilde{X}_n + \widehat{X}_n\Big) \\
&= U\Big(1 + r + \widetilde{X}_n + (1-y_n)(\widehat{\theta} \cdot S)_\tau\Big) \\
&\geq U\Big(1 + r - y_n + (1-y_n)(\widehat{\theta} \cdot S)_\tau\Big) \\
&= U\Big(y_n r + (1-y_n)\Big[1 + r + (\widehat{\theta} \cdot S)_\tau\Big]\Big) \\
&\geq y_n U(r) + (1-y_n) U\Big(1 + r + (\widehat{\theta} \cdot S)_\tau\Big).
\end{aligned}$$

This implies that $\widetilde{\theta}^{(n)} \in \mathcal{A}_{adm}(1+r, S^\tau)$. On one hand, a combination of the previous inequality and Fatou's Lemma implies that

$$\begin{aligned}
&E\Big\{U\left(1 + r + \widetilde{X} + (\widehat{\theta} \cdot S)_\tau\right) - U\left(1 + r + (\widehat{\theta} \cdot S)_\tau\right)\Big\} \\
&= E\Big\{\lim_n \Big[U\Big(1 + r + \widetilde{X}_n + \widehat{X}_n\Big) - (1-y_n)U\Big(1 + r + (\widehat{\theta} \cdot S)_\tau\Big) - y_n U(r)\Big]\Big\} \\
&= E\Big\{\lim_n \Big[U\Big(1 + r + (\widetilde{\theta}^{(n)} \cdot S)_\tau\Big) - (1-y_n)U\Big(1 + r + (\widehat{\theta} \cdot S)_\tau\Big) - y_n U(r)\Big]\Big\} \qquad (3.8) \\
&\leq \liminf_n E\Big\{U\Big(1 + r + (\widetilde{\theta}^{(n)} \cdot S)_\tau\Big) - (1-y_n)U\Big(1 + r + (\widehat{\theta} \cdot S)_\tau\Big) - y_n U(r)\Big\} \\
&\leq \liminf_n E\Big\{U\Big(1 + r + (\widehat{\theta} \cdot S)_\tau\Big) - (1-y_n)U\Big(1 + r + (\widehat{\theta} \cdot S)_\tau\Big) - y_n U(r)\Big\} = 0.
\end{aligned}$$

On the other hand, since $P(\widetilde{X} > 0) > 0$ and U is strictly increasing, we get

$$E\Big\{U\Big(1 + r + \widetilde{X} + (\widehat{\theta} \cdot S)_\tau\Big)\Big\} > E\Big\{U\Big(1 + r + (\widehat{\theta} \cdot S)_\tau\Big)\Big\}.$$

This is a contradiction with (3.8), and the NUPBR for S^τ is fulfilled. Thanks to Proposition 3.1, the global NUPBR for S follows immediately, and the proof of the theorem is completed. ∎

3.2 The case of possible negative wealth

In this section the utility function U, see Definition 2.8, satisfies

$$\mathrm{dom}(U) = \mathbb{R}, \quad U'(\infty) = 0, \quad U'(-\infty) = \infty, \tag{3.9}$$

and

$$AE_{+\infty}(U) := \limsup_{x\to+\infty} \frac{xU'(x)}{U(x)} < 1, \quad AE_{-\infty}(U) := \liminf_{x\to-\infty} \frac{xU'(x)}{U(x)} > 1. \tag{3.10}$$

These conditions are used by Schachermayer in [37], which are essential in his proofs. In the current setting, for the initial capital x, the set of admissible strategies will be denoted by $\Theta(x, S)$ and is given by

$$\Theta(x, S) := \Big\{ H \text{ admissible} \quad \Big| \quad E\Big[\Big|U(x + H \cdot S_T)\Big|\Big] < +\infty \Big\}. \tag{3.11}$$

Let us point out the remark in [37] that when the utility maximization problem is bounded by $U(\infty)$, the required integrability in the definition of $\Theta(x, S)$ is superfluous and in that case, we shall simply put $\Theta(S)$ for short.

Below, we state our version of the FTUM for the above class of utilities.

Theorem 3.3. *Let U be a utility function satisfying (3.9)–(3.10). Suppose there exist an increasing stopping times (T_n) that increases stationarily to T and x_n such that*

$$\sup_{\theta\in\Theta(x_n,S^{T_n})} EU\Big(x_n + (\theta \cdot S)_{T_n}\Big) < U(+\infty), \qquad \forall n.$$

Then the following properties are equivalent:
(1) There exist a sequence of stopping times $(\tau_n)_{n\geq1}$ that increases stationarily to T such that for any $n \geq 0$ and initial wealth x_0, there exist $W^* \in L^0$ and $(\widehat{\theta}^m)_{m\geq1} \in \Theta(x_0, S^{\tau_n})$ such that

$$\max_{\theta\in\Theta(x_0,S^{\tau_n})} EU\Big(x_0 + (\theta \cdot S)_{\tau_n}\Big) = EU(W^*) < U(+\infty),$$

and

$$\lim_m E\left[\Big|U(W^*) - U(x_0 + \widehat{\theta}^m \cdot S_{\tau_n})\Big|\right] = 0.$$

(2) S satisfies NUPBR.

Proof. We will start proving the easiest part which is (2) $\Longrightarrow$ (1). As in the proof of Theorem 3.2, we use Theorem 2.7 and conclude the existence of a local martingale density $Z > 0$, such that $Z, ZS \in \mathcal{M}_{loc}(P)$. Consider a localization sequence for Z and ZS that we denote by $(\sigma_n)_{n\geq1}$, then Z^{σ_n} and $Z^{\sigma_n}S^{\sigma_n}$ are both martingale under P. Put

$$Q_n := Z^{\sigma_n} \cdot P, \quad \tau_n := T_n \wedge \sigma_n \uparrow T.$$

Thus, Q_n is an equivalent martingale measure for S^{σ_n} and for S^{τ_n}. Since $\theta I_{[0,\tau_n]} \in \Theta(x_n, S^{T_n})$ whenever $\theta \in \Theta(x_n, S^{\tau_n})$, we get

$$\sup_{\theta\in\Theta(x_n,S^{\tau_n})} EU\Big(x_n + (\theta \cdot S)_{\tau_n}\Big) \leq \sup_{\varphi\in\Theta(x_n,S^{T_n})} EU\Big(x_n + (\varphi \cdot S)_{T_n}\Big) < U(+\infty), \qquad \forall n.$$

Thus by the Theorem 2.2 in [37], it follows that for any $n \geq 0$ and initial wealth x_0, there exist $W^* \in L^0$ and $\widehat{\theta}^m \in \Theta(x_0, S^{\tau_n})$ such that

$$\lim_m E\Big[\Big|U(W^*) - U(x_0 + \widehat{\theta}^m \cdot S_{\tau_n})\Big|\Big] = 0,$$

and

$$\max_{\theta \in \Theta(x_0, S^{\tau_n})} EU\Big(x_0 + (\theta \cdot S)_{\tau_n}\Big) = EU(W^*) < U(+\infty).$$

This proves the assertion (1). Now we focus on proving the reverse implication and start with assuming that assertion (1) holds. Without loss of generality, we take $x_0 > 1$, and due to

$$\lim_m E\Big[\Big|U(W^*) - U\Big(x_0 + (\widehat{\theta}^m \cdot S)_{\tau_n}\Big)\Big|\Big] \longrightarrow 0,$$

we obtain the convergence in probability of $U\Big(x_0 + (\widehat{\theta} \cdot S)_{\tau_n}\Big)$ to $U(W^*)$. Thus, $\Big(x_0 + (\widehat{\theta}^m \cdot S)\Big)_{\tau_n}$ converges in probability to W^* when m goes to infinity. By taking a subsequence, w.l.o.g., we assume that the sequence converges almost surely to W^*. Then, for any $\lambda \in (0,1)$ and any $\theta \in \Theta(x_0, S_{\tau_n})$, we derive

$$\begin{aligned} \lambda f(\lambda, m) := \Big\{ & U\Big(x_0 + (\widehat{\theta}^m \cdot S)_{\tau_n} + \lambda((\theta - \widehat{\theta}^m) \cdot S)_{\tau_n}\Big) \\ & -\lambda U(W^*) - (1-\lambda)U\Big(x_0 + (\widehat{\theta}^m \cdot S)_{\tau_n}\Big)\Big\} \\ \geq & \lambda\Big[U(x_0 + \theta \cdot S_{\tau_n}) - U(W^*)\Big]. \end{aligned} \tag{3.12}$$

Since the RHS term in the above inequality is integrable, then due to Fatou's Lemma, this implies that

$$\begin{aligned} & E[(x_0 + \theta \cdot S_{\tau_n} - W^*)U'(W^*)] = E[\underline{\lim}_{\lambda \downarrow 0} \lim_m f(\lambda, m)] \\ & \leq \underline{\lim}_{\lambda \downarrow 0} \lim_m \tfrac{1-\lambda}{\lambda}\Big\{EU(W^*) - EU(x_0 + (\widehat{\theta}^m \cdot S)_{\tau_n})\Big\} = 0. \end{aligned} \tag{3.13}$$

By combining the inequality $0 \leq \xi U'(\xi) \leq U(\xi) - U(0)$ (for any $\xi \in L^0_+(P)$) and (3.13), we obtain

$$\begin{aligned} & E[(x_0 + (\theta \cdot S)_{\tau_n})U'(W^*)] \leq E[W^* U'(W^*)] < +\infty, \text{ and} \\ & x_0 E[U'(W^*)] \leq E[W^* U'(W^*)] < +\infty. \end{aligned} \tag{3.14}$$

Consider the probability measure, denoted by R, and given by

$$R := \frac{U'(W^*)}{E[U'(W^*)]} \cdot P.$$

Then, (3.14) implies that

$$E^R\left[x_0 + \theta \cdot S_{\tau_n}\right] \leq E^R[W^*]. \tag{3.15}$$

Thus, the set $\{1 + (\theta \cdot S)_{\tau_n} \ \mid \ \theta \in L(S) \text{ and } (\theta \cdot S) \geq -1\}$ is a bounded set in $L^1(R)$. This implies the boundedness of this set in probability under any equivalent probability and the NUPBR for S^{τ_n} follows. Thanks to Proposition 3.1, we obtain the global NUPBR for S. This ends the proof of the theorem. ∎

3.3 The case of exponential utility

Even though the exponential utility is a particular case of Subsection 3.2, it deserves special attention for two reasons. The first reason lies in the popularity of the exponential utility, while the second reason lies in our belief that for this case, when S is locally bounded, we may obtain more precise results with less assumptions. Through out this section the set of admissible strategies for the model (X, Q) will be denoted by $\Theta(X, Q)$, and is given by

$$\Theta(X, Q) := \left\{ \theta \in L(X) \mid \ E^Q\left(e^{-\theta \cdot X_T}\right) < +\infty \right\}.$$

When $Q = P$, we simply write $\Theta(X) := \Theta(X, P)$ for short. Then, the set of local martingale densities that are locally in $L \log L$ will be denoted by

$$\mathcal{Z}^e_{f,loc}(X, Q) := \left\{ Z > 0 \middle| \ Z,\ ZX \in \mathcal{M}_{loc}(Q),\ Z\log(Z) \text{ is } Q\text{-locally integrable} \right\}. \tag{3.16}$$

when $Q = P$, we simply write $\mathcal{Z}^e_{f,loc}(X)$.

Definitions 3.4. *Let $Z = \mathcal{E}(N) \geq 0$, where $N \in \mathcal{M}_{0,loc}(P)$. Then $Z\log(Z)$ is locally integrable if and only if*

$$V^{(E)}(N) := \frac{1}{2}\langle N^c\rangle + \sum \Big[(1 + \Delta N)\log(1 + \Delta N) - \Delta N\Big], \tag{3.17}$$

belongs to $\mathcal{A}^+_{loc}(P)$. In this case, its compensator is called the entropy-Hellinger process of Z and is denoted by $h^{(E)}(Z, P)$ (see [8] for details).

Theorem 3.5. *Suppose S is locally bounded. Then the following are equivalent:*
(1) There exist a sequence of stopping times $(\tau_n)_{n\geq 1}$ increasing stationarily to T and $\widehat{\theta}^n \in \Theta(S^{\tau_n})$ such that

$$\min_{\theta \in \Theta(S^{\tau_n})} E\left(e^{-\theta \cdot S_{\tau_n}}\right) = E\left(e^{-\widehat{\theta}^n \cdot S_{\tau_n}}\right), \quad \forall\, n \geq 1. \tag{3.18}$$

(2) $\mathcal{Z}^e_{f,loc}(S) \neq \emptyset$.

Proof. (i) In this part, we will prove (2)$\Longrightarrow$(1), for which we will use Theorem 2.1 of [29]. Let $(\tau_n)_{n\geq 1}$ be a sequence of stopping times that increases stationarily to T such that Z^{τ_n} is a true martingale and $Z_{\tau_n}\log(Z_{\tau_n})$ is integrable. Then S^{τ_n} admits an equivalent local martingale measure $Q^n := Z_{\tau_n} \cdot P$ with finite entropy. Therefore, Theorem 2.1 of [29] implies the existence of an optimal solution $\widehat{\theta}^n \in L(S^{\tau_n})$ for (3.18). This proves assertion (2).
(ii) In the remaining part of this proof, we will prove (1)$\Longrightarrow$(2). Suppose that assertion (1) holds. Since the null strategy $0 \in \Theta(S^{\tau_n})$, then we obtain

$$E(e^{-\widehat{\theta}^n \cdot S_{\tau_n}}) \leq E(e^{-0 \cdot S_{\tau_n}}) = 1,$$

and the integrability of $e^{-\widehat{\theta}^n \cdot S_{\tau_n}}$ follows. Put

$$Q^n := \frac{\exp(-\widehat{\theta}^n \cdot S_{\tau_n})}{E\left(\exp(-\widehat{\theta}^n \cdot S_{\tau_n})\right)} \cdot P,$$

and consider a sequence of stopping times $(T_n)_{n\geq 1}$ that increases stationarily to T and S^{T_n} is bounded. For any bounded and predictable process θ, we have $\pm\theta I_{]\!]0,T_n]\!]} \in \Theta(S^{\tau_n})$ as well as

$$\phi_\lambda := -\lambda\theta I_{]\!]0,T_n]\!]} + \widehat{\theta}^n \in \Theta(S^{\tau_n}), \qquad \text{for any } \ \lambda \in (0, 1).$$

This implies that $E\left(e^{-\phi_\lambda\cdot S_{\tau_n}}-e^{-\widehat{\theta}^n\cdot S_{\tau_n}}\right)\geq 0$, or equivalently

$$E^{Q^n}\left(e^{\lambda\theta\cdot S_{\tau_n\wedge T_n}}-1\right)\geq 0,\quad \forall\,\lambda\in(0,1). \tag{3.19}$$

Then, by combining concavity (that implies that $1-e^{\lambda\theta\cdot S_{\tau_n\wedge T_n}}\geq\lambda(1-e^{\theta\cdot S_{\tau_n\wedge T_n}})$), Fatou's Lemma and (3.19), we derive

$$E^{Q^n}(\theta\cdot S_{\tau_n\wedge T_n})\leq\lim_{\lambda\to 0}E^{Q^n}\left(\frac{1-e^{\lambda\theta\cdot S_{\tau_n\wedge T_n}}}{\lambda}\right)\leq 0. \tag{3.20}$$

Since θ is an arbitrary bounded and predictable process, the above inequality holds for $-\theta$ also, and hence we conclude that Q_n is a martingale measure for S^{σ_n}, where $\sigma_n:=T_n\wedge\tau_n$. The density process of this martingale measure will be denoted by

$$\widehat{Z}^n_t:=\frac{E(\exp(-\widehat{\theta}^n\cdot S_{\tau_n})|\mathcal{F}_t)}{E\left(\exp(-\widehat{\theta}^n\cdot S_{\tau_n})\right)}=:\mathcal{E}_t\left(\widehat{N}^{(n)}\right).$$

For any $\theta\in\Theta(S^{\tau_n})$, and any $\lambda\in(0,1)$ put $\phi:=\lambda\theta+(1-\lambda)\widehat{\theta}\in\Theta(S^{\tau_n})$. Then in one hand the convexity of the exponential leads to conclude that $(\theta\cdot S^{\tau_n}-\widehat{\theta}\cdot S^{\tau_n})\exp(-\widehat{\theta}\cdot S^{\tau_n})$ is bounded from below by an integrable random variable. On the other hand, again the convexity of e^x combined with Fatou's lemma imply that

$$E\left(e^{-\widehat{\theta}\cdot S_{\tau_n}}(\theta-\widehat{\theta})\cdot S_{\tau_n}\right)\leq 0.$$

Then, it is easy to check that $(\widehat{\theta}\cdot S)_{\tau_n}\exp\left[-(\widehat{\theta}\cdot S)_{\tau_n}\right]$ is an integrable random variable. By combining this with

$$\widehat{Z}^n_{\tau_n}\log(\widehat{Z}^n_{\tau_n})=\frac{-\widehat{\theta}^n\cdot S_{\tau_n}\exp(-\widehat{\theta}^n\cdot S_{\tau_n})-\exp(-\widehat{\theta}^n\cdot S_{\tau_n})\log\left(E\left(\exp(-\widehat{\theta}^n\cdot S_{\tau_n})\right)\right)}{E\left(\exp(-\widehat{\theta}^n\cdot S_{\tau_n})\right)},$$

we deduce that $\widehat{Z}^n_{\tau_n}\log(\widehat{Z}^n_{\tau_n})$ is integrable, and hence $\widehat{Z}^n$ is a martingale density for S^{σ_n} that is $L\log L$-integrable. Then, by putting

$$\widehat{N}:=\sum_{n=1}^{+\infty}I_{]\!]\sigma_{n-1},\sigma_n]\!]}\cdot\widehat{N}^{(n)},$$

and applying Lemma 3.6, assertion (ii) follows immediately. This ends the proof of the theorem. ☐

Lemma 3.6. *Let $(\tau_n)_{n\geq 1}$ be a sequence of stopping times that increases stationarily to T, and $(N^{(n)})$ be a sequence of local martingale. Then, the process*

$$N:=\sum_{n=1}^{+\infty}I_{]\!]\tau_{n-1},\tau_n]\!]}\cdot N^{(n)},\quad (\tau_0=0),$$

is a local martingale satisfying:
(i) If $\mathcal{E}(N^{(n)})>0$ for any $n\geq 1$, then $\mathcal{E}(N)>0$.
(ii) If $V^{(E)}(N^{(n)})\in\mathcal{A}^+_{loc}(P)$ for any $n\geq 1$, then $V^{(E)}(N)\in\mathcal{A}^+_{loc}(P)$.
(iii) If $\mathcal{E}(N^{(n)})$ is a σ-martingale density for S^{τ_n} for any $n\geq 1$, then $\mathcal{E}(N)$ is a σ-martingale density for S.

Proof. It is obvious that

$$N^{\tau_n} = \sum_{k=1}^{n} I_{]\!]\tau_{k-1},\tau_k]\!]} \cdot N^{(k)} \in \mathcal{M}_{0,loc}(P).$$

This proves that $N \in (\mathcal{M}_{0,loc}(P))_{loc} = \mathcal{M}_{0,loc}(P)$, and $\mathcal{E}(N) > 0$ which is due to

$$1 + \Delta N = 1 + \Delta N^{(n)} > 0 \quad \text{on} \quad]\!]\tau_{n-1}, \tau_n]\!], \qquad n \geq 1.$$

Then, due to the definition of the operator $V^{(E)}$ given by (3.17), it is also easy to remark that $V^{(E)}(I_{]\!]\sigma,\tau]\!]} \cdot M) = I_{]\!]\sigma,\tau]\!]} \cdot V^{(E)}(M)$ for any local martingale M (with $1 + \Delta M \geq 0$) and any pair of stopping times τ and σ such that $\tau \geq \sigma$. Thus, we get

$$\left(V^{(E)}(N)\right)^{\tau_n} = \sum_{k=1}^{n} I_{]\!]\tau_{k-1},\tau_k]\!]} \cdot V^{(E)}(N^{(k)}) \in \mathcal{A}^+_{loc}(P).$$

Hence, we deduce (thanks to Lemma 2.1) that $V^{(E)}(N) \in \left(\mathcal{A}^+_{loc}(P)\right)_{loc} = \mathcal{A}^+_{loc}(P)$. This ends the proof of assertion (i) and (ii) of the lemma. To prove the last assertion, we first remark that $\mathcal{E}(M)$ is a σ-martingale density for S if and only if there exists a predictable process φ such that $0 < \varphi \leq 1$ and

$$\varphi \cdot S + \varphi \cdot [S, M] \in \mathcal{M}_{0,loc}(P).$$

Therefore, since $\mathcal{E}(N^{(n)})$ is a σ-martingale density for S^{τ_n} for each $n \geq 1$, then there exists ϕ_n such that $0 < \phi_n \leq 1$ and

$$Y_n := \phi_n \cdot S + \phi_n \cdot [S^{\tau_n}, N^{(n)}] \in \mathcal{M}_{0,loc}(P), \qquad \forall \;\; n \geq 1. \tag{3.21}$$

Put $\phi := \sum_{k=1}^{+\infty} I_{]\!]\tau_{k-1},\tau_k]\!]}\phi_k$. Thus, it is easy to prove that $0 < \phi \leq 1$, and

$$(\phi \cdot S + \phi \cdot [S, N])^{\tau_n} = \sum_{k=1}^{n} I_{]\!]\tau_{k-1},\tau_k]\!]} \cdot Y_k \in \mathcal{M}_{0,loc}(P).$$

Hence, due to Lemma 2.1 again, we deduce that

$$\phi \cdot S + \phi \cdot [S, N] \in \mathcal{M}_{0,loc}(P),$$

which is equivalent to the fact that $\mathcal{E}(N)$ is a σ-martingale density for S. This ends the proof of the lemma. ☐

4 Application: Numéraire Portfolio/NUPBR/Weak Viability

We start by recalling the definition of the numéraire portfolio.

Definitions 4.1. *A process $\widetilde{X} \in \mathcal{X}(x_0)$ is called a numéraire portfolio under P if for every $X \in \mathcal{X}(x_0)$, the relative wealth process $X/\widetilde{X}$ is a supermartingale .*

It is easy to prove that the numéraire portfolio –when it exists– is unique. For other properties or more details about the numéraire portfolio, we refer the reader to [5], [31], [39] and the references therein. Below, we will prove directly the stability of the numéraire portfolio under the localization.

Proposition 4.2. *Let $(\tau_n)_{n\geq 1}$ be a sequence of stopping times increasing stationarily to T. If for each n, the stopped process S^{τ_n} admits a numéraire portfolio, then S also admits a numéraire portfolio.*

Proof. For each n, let $W^{(n)}$ denote the numéraire portfolio for S^{τ_n}. Then, there exists a 1-admissible strategy, $\widehat{\theta}^n$, for S^{τ_n} such that $W^{(n)} = 1 + (\widehat{\theta}^n \cdot S^{\tau_n}) > 0$, and

$$\frac{1 + (\theta \cdot S)^{\tau_n}}{1 + (\widehat{\theta}^n \cdot S)^{\tau_n}}, \tag{4.22}$$

is a supermartingale under P for any 1-admissible strategy θ. In particular for $\theta = 0$, we use Fatou's Lemma and deduce that $E\left[\left(1 + (\widehat{\theta}^n \cdot S^{\tau_n})_{t-}\right)^{-1}\right] \leq 1$ for any $0 < t \leq T$. This implies that $1 + (\widehat{\theta}^n \cdot S)_- > 0$ and hence

$$\pi^n_t := \frac{\widehat{\theta}^n_t}{1 + \widehat{\theta}^n \cdot S^{\tau_n}_{t-}} \in L(S^{\tau_n}) \quad \text{and} \quad 1 + (\widehat{\theta}^n \cdot S)^{\tau_n} = \mathcal{E}\left((\pi^n \cdot S)^{\tau_n}\right).$$

Put $\tau_0 = 0$ and

$$\pi := \sum_{i \geq 1} \pi^i I_{]\!]\tau_{i-1}, \tau_i]\!]}. \tag{4.23}$$

Then, due to Theorem 9.19 in [24], we deduce that $\pi \in L(S)$ and $\mathcal{E}(\pi \cdot S) > 0$. Next, we will show that $\mathcal{E}(\pi \cdot S)$ is the numéraire portfolio for S. To this end, it is enough –due to Fatou's Lemma again – to prove that

$$\frac{\mathcal{E}(\theta \cdot S)}{\mathcal{E}(\pi \cdot S)} \text{ is a local supermartingale,} \tag{4.24}$$

for any $\theta \in L(S)$ such that $1 + \theta^T \Delta S > 0$. To prove this fact, we first remark that for any stopping time $\tau \leq T$, we have $\mathcal{E}(\pi \cdot S)_{\tau \wedge \tau_1} = \mathcal{E}(\widehat{\pi}^1 \cdot S)_{\tau \wedge \tau_1}$ and

$$\mathcal{E}(\pi \cdot S)_{\tau \wedge \tau_{n+1}} = \prod_{i=1}^{n+1} \frac{\mathcal{E}(\widehat{\pi}^i \cdot S)_{\tau \wedge \tau_i}}{\mathcal{E}(\widehat{\pi}^i \cdot S)_{\tau \wedge \tau_{i-1}}} = \mathcal{E}(\pi \cdot S)_{\tau \wedge \tau_n} \frac{\mathcal{E}(\widehat{\pi}^{n+1} \cdot S)_{\tau \wedge \tau_{n+1}}}{\mathcal{E}(\widehat{\pi}^{n+1} \cdot S)_{\tau \wedge \tau_n}} \qquad n \geq 1.$$

Therefore, for any stopping time $\tau \leq T$, we calculate

$$\begin{aligned} E\left[\frac{\mathcal{E}(\theta \cdot S)_{\tau \wedge \tau_{n+1}}}{\mathcal{E}(\pi \cdot S)_{\tau \wedge \tau_{n+1}}} \Big| \mathcal{F}_{\tau \wedge \tau_n}\right] &= E\left[\frac{\mathcal{E}(\widehat{\pi}^{n+1} \cdot S)_{\tau \wedge \tau_n} \mathcal{E}(\theta \cdot S)_{\tau \wedge \tau_{n+1}}}{\mathcal{E}(\pi \cdot S)_{\tau \wedge \tau_n} \mathcal{E}(\widehat{\pi}^{n+1} \cdot S)_{\tau \wedge \tau_{n+1}}} \Big| \mathcal{F}_{\tau \wedge \tau_n}\right] \\ &= \frac{\mathcal{E}(\widehat{\pi}^{n+1} \cdot S)_{\tau \wedge \tau_n}}{\mathcal{E}(\pi \cdot S)_{\tau \wedge \tau_n}} E\left[\frac{\mathcal{E}(\theta \cdot S)_{\tau \wedge \tau_{n+1}}}{\mathcal{E}(\widehat{\pi}^{n+1} \cdot S)_{\tau \wedge \tau_{n+1}}} \Big| \mathcal{F}_{\tau \wedge \tau_n}\right] \\ &\leq \frac{\mathcal{E}(\theta \cdot S)_{\tau \wedge \tau_n}}{\mathcal{E}(\pi \cdot S)_{\tau \wedge \tau_n}}. \end{aligned}$$

Hence, by iteration and using Fatou's Lemma afterwards, we derive

$$E\left[\frac{\mathcal{E}(\theta \cdot S)_\tau}{\mathcal{E}(\pi \cdot S)_\tau}\right] \leq \liminf_n E\left[\frac{\mathcal{E}(\theta \cdot S)_{\tau \wedge \tau_n}}{\mathcal{E}(\pi \cdot S)_{\tau \wedge \tau_n}}\right] \leq E\left[\frac{\mathcal{E}(\theta \cdot S)_{\tau \wedge \tau_1}}{\mathcal{E}(\widehat{\pi}^1 \cdot S)_{\tau \wedge \tau_1}}\right] \leq 1.$$

Thus, a direct application of Lemma 4.4 (see at the end of this section) implies that $\mathcal{E}(\pi \cdot S)$ is a numéraire portfolio for S, and the proof of the proposition is achieved. ∎

Below we state the main result of this section.

Theorem 4.3. *The following properties are equivalent:*
(i) S satisfies NUPBR.
(ii) The set $\mathcal{Z}_{loc}(S)$ (defined in (2.2)) is not empty.

(iii) There exists a probability $Q \sim P$, such that for any utility U satisfying (3.5) and $x \in dom(U)$, there exists $\widehat{\theta} \in \mathcal{A}_{adm}(x, U, S, Q)$ such that

$$\max_{\theta \in \mathcal{A}_{adm}(x,U,S,Q)} E^Q U\Big(x + (\theta \cdot S)_T\Big) = E^Q U\left(x + (\widehat{\theta} \cdot S)_T\right) < +\infty. \tag{4.25}$$

(iv) For any $\epsilon > 0$, there exists $Q_\epsilon \sim P$ such that $E|\frac{dQ_\epsilon}{dP} - 1| \leq \epsilon$, and for any utility U satisfying (3.5) and $x \in dom(U)$,there exists $\widetilde{\theta}_\epsilon \in \mathcal{A}_{adm}(x, U, S, Q_\epsilon)$ satisfying

$$\max_{\theta \in \mathcal{A}_{adm}(x,U,S,Q_\epsilon)} E^{Q_\epsilon} U\Big(x + (\theta \cdot S)_T\Big) = E^{Q_\epsilon} U\left(x + (\widetilde{\theta}_\epsilon \cdot S)_T\right) < +\infty. \tag{4.26}$$

(v) For any $\epsilon \in (0, 1)$ there exist $Q_\epsilon \sim P$ and $\widetilde{\theta}_\epsilon \in \mathcal{A}_{\epsilon,1} := \mathcal{A}_{adm}(1, \log, S, Q_\epsilon)$ such that $E|\frac{dQ_\epsilon}{dP} - 1| \leq \epsilon$ and

$$\max_{\theta \in \mathcal{A}_{\epsilon,1}} E^{Q_\epsilon} \log\Big(1 + (\theta \cdot S)_T\Big) = E^{Q_\epsilon} \log\left(1 + (\widetilde{\theta}_\epsilon \cdot S)_T\right) < +\infty. \tag{4.27}$$

(vi) The numéraire portfolio for S exists.

Remark: This theorem completely establishes the connection between the NUPBR, existence of solution to the log-utility maximization, existence of solution to utility maximization for any utility satisfying (3.5), and existence of the numéraire portfolio. To describe precisely the novelty of this theorem, we need to recall the existing literature related in a way or another to our theorem. The equivalence between (i) and (vi) of the theorem was established in [31]. Our proof is very different, sounds shorter, and less technical (it does not use semimartingale characteristics that are very powerful tools but not easy to handle). This equivalence between (i) and (vi) was also investigated by Becherer [5] and Christensen and Larsen [9]. These authors also explored the connection of (i) (or equivalently (vi)) to the existence of growth-optimal portfolio and the existence of solution to log-utility maximization. A very nice summary of these results can be found in the recent paper of Hulley and Schweizer (see Theorem 2.3. of [22]). Precisely, this theorem claims the equivalence between (1), (2), and (3) below.
(1) S satisfies NUPBR.
(2) The numéraire portfolio X^{np} exists.
(3) The growth-optimal portfolio X^{go} exists.
If furthermore

$$\sup\Big\{\ E\Big[\log X_T\Big]\ \Big|\ X \in \mathcal{X}(1), \quad X_- > 0, \quad \text{and} \quad E\left[(\log X_T)^-\right] < \infty\Big\} < \infty,$$

then the properties (1), (2), and (3) are also equivalent to the existence of the solution of the log-utility maximization. Our Theorem 4.3 clearly states the equivalence between all the four properties without any assumption and for any utility satisfying (3.5), but with the appropriate formulation. This formulation uses the appropriate change of probability. More importantly, we show that this probability can be chosen as close as we want to P (but may not be equal to P of course). This can be seen a sort of *weak* viability. In economics, viability holds when one can find a utility (in larger sense than Definition 2.8) and an initial capital such that corresponding utility-maximization admits solution under the original probability measure P. We call a weakly viable market if there is an agent with utility as in Definition 2.8 and satisfies (3.5) and an initial capital for which the optimal portfolio exists under an equivalent probability (under an equivalent subjective belief than the real probability measure). Then, our theorem establishes the equivalence between the weak viability, NUPBR and existence of numéraire portfolio.

Proof. of Theorem 4.3: The proof of this theorem will be achieved after three steps. The first step will focus on proving $(i) \iff (ii) \iff (iii)$. The second step will prove $(i) \iff (iv) \iff (v)$, while the last step will deal with with $(v) \Longrightarrow (vi) \Longrightarrow (i)$.
1) The equivalence $(i) \iff (ii)$ is exactly Takaoka's result (see Theorem 2.7).
The proof of $(i) \iff (iii)$ boils down to the proof of $(i) \Longrightarrow (iii)$, since the reverse implication follows directly from Theorem 3.2. Suppose that assertion (i) holds and due to the equivalence of (i) and (ii), we consider $Z \in \mathcal{Z}_{loc}(S)$ (i.e. a σ-martingale density for S). Then put

$$Q := \frac{Z_T}{E\left[Z_T\right]} \cdot P \sim P. \tag{4.28}$$

Let U be a utility function satisfying (3.5) and $x \in \mathrm{dom}(U)$. Thus, in virtue of Remark **??**, for any $\theta \in \mathcal{A}_{adm}(x, U, S, Q)$, $Z(x + \theta \cdot S)$ is a nonnegative local martingale, and hence a supermartingale. Then, by the concavity of U, it is easy to see that

$$\sup_{\theta \in \mathcal{A}_{adm}(x,U,S,Q)} E^Q U\left(x + (\theta \cdot S)_T\right) \leq U(x/E\left[Z_T\right]) < +\infty.$$

Therefore, a direct application of Theorem 3.2 under Q implies the existence of a sequence of stopping times $(\tau_n)_{n\geq 1}$ that increases stationarily to T and a sequence $\widehat{\theta}^{(n)} \in \mathcal{A}_{adm}(x, U, S^{\tau_n}, Q)$ such that

$$\sup_{\theta \in \mathcal{A}_{adm}(x,U,S^{\tau_n},Q)} E^Q U\left(x + (\theta \cdot S)_{\tau_n}\right) = E^Q U\left(x + (\widehat{\theta}^{(n)} \cdot S)_{\tau_n}\right). \tag{4.29}$$

Due to Takaoka's result (see Theorem 2.7), we deduce that $\mathcal{Z}_{loc}(S) \neq \emptyset$, and hence a direct application of Corollary 2.12, we deduce the existence of $(a_l)_{l\geq 1}$ ($a_l \in (0,1)$), $\widehat{\theta} \in L(S)$, and a nondecreasing RCLL process C such that $C_0 = 0$,

$$\sum_{l=n}^{m_n} a_l = 1, \text{ and } x + \sum_{l=n}^{m_n} a_l \widehat{\theta}^{(l)} \cdot S^{\tau_l} \quad \text{Fatou converges to} \quad x + \widehat{\theta} \cdot S - C. \tag{4.30}$$

Hence, the proof of assertion (iii) will be completed if we prove $\widehat{\theta}$ belongs to $\mathcal{A}_{adm}(x, U, S, Q)$ and it is optimal solution to (4.25). We start by proving the admissibility of $\widehat{\theta}$. Due to Fatou's lemma and the concavity of U, we get

$$\begin{aligned} E^Q U^-(x + \widehat{\theta} \cdot S_T) &\leq \liminf E^Q U^-(x + \textstyle\sum_{l=n}^{m_n} a_l \widehat{\theta}^{(l)} \cdot S_{\tau_l}) \\ &\leq \liminf \textstyle\sum_{l=n}^{m_n} a_l E^Q U^-(x + \widehat{\theta}^{(l)} \cdot S_{\tau_l}). \end{aligned} \tag{4.31}$$

If $U(\infty) \leq 0$, then we have

$$\sum_{l=n}^{m_n} a_l E^Q U^-(x + \widehat{\theta}^{(l)} \cdot S_{\tau_l}) = -\sum_{l=n}^{m_n} a_l E^Q U(x + \widehat{\theta}^{(l)} \cdot S_{\tau_l}) \leq -U(x) < +\infty,$$

and the admissibility of $\widehat{\theta}$ follows immediately from this inequality and (4.31). Suppose that $U(+\infty) > 0$. Then, there exists a real number r such that $U(r) > 0$, and the following hold

$$\begin{aligned} &\liminf \sum_{l=n}^{m_n} a_l E^Q U^-(x + \widehat{\theta}^{(l)} \cdot S_{\tau_l}) \\ &\leq \liminf \sum_{l=n}^{m_n} a_l E^Q U(r + x + \widehat{\theta}^{(l)} \cdot S_{\tau_l}) - U(x) \\ &\leq U\left(\tfrac{r+x}{E[Z_T]}\right) - U(x) < +\infty. \end{aligned} \tag{4.32}$$

This combined with (4.31) completes the proof of $\widehat{\theta} \in \mathcal{A}_{adm}(x, S, Q)$. Furthermore, we get $U(x + \widehat{\theta} \cdot S_T) \in L^1(Q)$. Next, we will prove the optimality of the strategy $\widehat{\theta}$. To this end, we will start by proving

$$E^Q U(x + \widehat{\theta} \cdot S_T) \geq \limsup E^Q U(x + \sum_{l=n}^{m_n} a_l \widehat{\theta}^{(l)} \cdot S_{\tau_l}). \tag{4.33}$$

If $U(+\infty) \leq 0$, then the above inequality follows immediately from Fatou's lemma. Suppose that $U(+\infty) > 0$. In this case, by mimicking the proof of Lemma 3.2 of [27], we easily prove that

$$\left\{ U(y_n) : \quad n \geq 1, \quad y_n := x + \sum_{l=n}^{m_n} a_l \widehat{\theta}^{(l)} \cdot S_{\tau_l} \right\} \text{ is } Q\text{-uniformaly integrable.} \tag{4.34}$$

Denote the inverse of U by $\phi : (U(0+), U(+\infty)) \to (0, +\infty)$. Then we derive $E^Q[\phi(U(y_n))] \leq x/E(Z_T)$ and due to l'Hospital rule and (3.5) we have

$$\lim_{x \to U(+\infty)} \frac{\phi(x)}{x} = \lim_{y \to +\infty} \frac{y}{U(y)} = \lim_{y \to +\infty} \frac{1}{U'(y)} = +\infty.$$

Then, the uniform integrability of the sequence $(U(y_n))_{n \geq 1}$ follows from the La-Vallée-Poussin argument. Then, (4.33) follows immediately from this uniform integrability and (4.30). Therefore, we obtain

$$\begin{aligned}
E^Q U(x + \widehat{\theta} \cdot S_T) &\geq \limsup E^Q U(x + \sum_{l=n}^{m_n} a_l \widehat{\theta}^{(l)} \cdot S_{\tau_l}) \\
&\geq \limsup \sum_{l=n}^{m_n} a_l E^Q U(x + \widehat{\theta}^{(l)} \cdot S_{\tau_l}) \\
&\geq \limsup \sum_{l=n}^{m_n} a_l E^Q U(x + \epsilon\theta \cdot S_{\tau_l}) \qquad (4.35) \\
&\geq \liminf \sum_{l=n}^{m_n} a_l E^Q U(x + \epsilon\theta \cdot S_{\tau_l}) \\
&\geq E^Q U(x + \epsilon\theta \cdot S_T) \qquad (4.36) \\
&\geq (1 - \varepsilon) U(x) + \varepsilon E^Q U(x + \theta \cdot S_T),
\end{aligned}$$

for any $\theta \in \mathcal{A}_{adm}(x, S, Q)$, and any $\epsilon \in (0, 1)$. It is clear that the optimality of $\widehat{\theta}$ follows immediately from the above inequalities by taking ϵ to one. It is obvious that (4.35) follows from (4.29), while (4.36) follows from Fatou's lemma and $U(x + \epsilon(\theta \cdot S)_{\tau_n}) \geq U((1 - \epsilon)x) > -\infty$. This proves assertion (iii), and the proof of $(i) \iff (iii)$ is achieved.

2) The proof of $(i) \iff (iv)$, can be obtained by following exactly the same arguments as in the proof of $(i) \iff (iii)$ as long as we find Q_δ equivalent to P whose density converges to one in $L^1(P)$ when δ goes to zero, and for any $\delta > 0$, Q_δ has the same features used in the proof of $(i) \iff (iii)$. To prove that this latter claim holds, we consider $\delta \in (0, 1)$ and define

$$q := \frac{Z_T}{E[Z_T]}, \quad q_\delta := \frac{q}{\delta + q}, \quad Z_\delta := \frac{q_\delta}{E[q_\delta]} := q_\delta C_\delta, \quad Q_\delta := Z_\delta \cdot P \sim P. \tag{4.37}$$

It is very easy to check (4.25) has a solution under this Q_δ for any $\delta > 0$, and this probability has the same features as Q of part 1). Then, due to

$$1 > (C_\delta)^{-1} = E\left(\frac{q}{\delta + q}\right) \geq E\left[\frac{q}{1+q}\right] =: \Delta_0,$$

we deduce that Z_δ is positive, bounded by $(\Delta_0)^{-1}$, and converges almost surely to one when δ goes to zero. Then, the dominated convergence theorem allows us to conclude that for any $\epsilon > 0$ there exists $\delta := \delta(\epsilon) > 0$ such that $E|Z_{\delta(\epsilon)} - 1| < \epsilon$. This ends the proof of the equivalence $(i) \Longleftrightarrow (iv)$.

The equivalence of $(i) \Longleftrightarrow (v)$, can again be easily proved the same way as the proof of $(i) \Longleftrightarrow (iv)$ by taking the log utility.

3) In the remaining part of this proof, we will prove $(v) \Longrightarrow (vi) \Longrightarrow (i)$. Suppose that assertion (v) holds (and hence we have $\mathcal{Z}_{loc}(S) \neq \emptyset$). Then, it is easy to see that assertion (v) implies the existence of numéraire portfolio under each Q_ϵ. Then, we deduce that for any $n \geq 1$, there exists $0 < Z_n = C_n q_n \in L^1(P)$ (here $q_n = \frac{n}{n+q^{-1}}$ where q is given by (4.37)), converges to one in $L^1(P)$ and W_n a numéraire portfolio for S under $Q_n := Z_n \cdot P$.
Then, a direct application of Corollary 2.12, we deduce the existence of $(a_n)_{n\geq 1}$ ($a_n \in (0,1)$), $\widetilde{\theta} \in L(S)$, and nondecreasing process C such that $C_0 = 0$ and

$$\sum_{k=n}^{m_n} a_l = 1 \text{ and } \sum_{k=n}^{m_n} a_k W_k \ \text{ is Fatou convegent to } \ \widetilde{W} = x + \widetilde{\theta}\cdot S - C =: \widehat{W} - C.$$

Let $W \in \mathcal{X}(x)$ be a wealth process, $b \in (0,1)$, $\alpha > 1$, and τ be a stopping time. Then, there exists a sequence of stopping times $(\tau_k)_{k\geq 1}$ that decreases to τ and takes values in $\left(\mathbb{Q}^+ \cap [0,T[\right) \cup \{T\}$ such that

$$\text{on } \quad \{\tau < T\} \qquad T \geq \tau_k > \tau, \qquad \text{and on } \{\tau = T\} \qquad \tau_k = T.$$

Then, due to Fatou's Lemma, we get

$$E\left(\tfrac{W_\tau}{\widehat{W}_\tau} \wedge \alpha\right) \leq E\left(\tfrac{W_\tau}{\widetilde{W}_\tau} \wedge \alpha\right) \leq \liminf_n \liminf_k E\left(\frac{W_{\tau_k}}{\sum_{l=n}^{m_n} a_l W_l(\tau_k)} \wedge \alpha\right)$$

$$\leq \liminf_n \liminf_k E\Big(\left[\sum_{l=n}^{m_n} a_l \frac{W_{\tau_k}}{W_l(\tau_k)}\right] \wedge \alpha\Big).$$

Since $q_n := \frac{n}{n+q^{-1}}$ is increasing in n, then for any $l \geq n$ and any k we have

$$\{E(q_n|\mathcal{F}_{\tau_k}) > b\} \subset \{E(q_l|\mathcal{F}_{\tau_k}) > b\} = \{1 < b^{-1}\frac{Z_l(\tau_k)}{C_l} = E(q_l|\mathcal{F}_{\tau_k})b^{-1}\}.$$

Hence, we derive

$$E\Big(\Big[\sum_{l=n}^{m_n} a_l \frac{W_{\tau_k}}{W_l(\tau_k)}\Big] \wedge \alpha\Big) = E\Big(\Big[\sum_{l=n}^{m_n} a_l \frac{W_{\tau_k}}{W_l(\tau_k)}\Big] \wedge \alpha I_{\{E(q_n|\mathcal{F}_{\tau_k})\le b\}}\Big) +$$

$$+ E\Big(\Big[\sum_{l=n}^{m_n} a_l \frac{W_{\tau_k}}{W_l(\tau_k)}\Big] \wedge \alpha I_{\{E(q_n|\mathcal{F}_{\tau_k})> b\}}\Big)$$

$$\le \alpha P\Big(E(q_n|\mathcal{F}_{\tau_k}) \le b\Big) + b^{-1} E\Big(\sum_{l=n}^{m_n} a_l \frac{Z_l(\tau_k)}{C_l}\frac{W_{\tau_k}}{W_l(\tau_k)}\Big)$$

$$\le \alpha P\Big(E(q_n|\mathcal{F}_{\tau_k}) \le b\Big) + b^{-1} \sum_{l=n}^{m_n} \frac{a_l}{C_l}.$$

Since both C_n and q_n converge to one when n goes to infinity, and $E(q_n|\mathcal{F}_{\tau_k})$ converges to $E(q_n|\mathcal{F}_\tau)$ when k goes to infinity, then it is obvious that

$$\alpha P\Big(E(q_n|\mathcal{F}_{\tau_k}) \le b\Big) + b^{-1} \sum_{l=n}^{m_n} \frac{a_l}{C_l} \quad \text{converges to} \quad b^{-1}$$

when k and afterwards n goes to infinity. Hence, we deduce that

$$E\left(\frac{W_\tau}{\widehat{W}_\tau} \wedge \alpha\right) \le b^{-1}$$

for any $b \in (0,1)$, $\alpha > 1$, and any stopping time τ. Thus, by taking b to one and α to $+\infty$ and using Fatou's Lemma we deduce that

$$E\left(\frac{W_\tau}{\widehat{W}_\tau}\right) \le 1.$$

A straightforward application of Lemma 4.4 (see at the end of this proof) leads to the conclusion that $\widehat{W}$ is a portfolio numéraire under P for S. The proof of assertion (vi) is completed.

The proof of the remaining implication (i.e. $(vi) \Longrightarrow (i)$) is easy, and will detailed below for the sake of completeness. Suppose that there exists a numéraire portfolio for (S, P) that we denoted by W^*. Then, for each $\theta \in \mathcal{X}(1)$,

$$\frac{1 + \theta \cdot S}{W^*} \quad \text{is a nonnegative supermartingale.}$$

As a result, for all $c > 0$, we obtain

$$P\Big(\frac{1 + (\theta \cdot S)_T}{W_T^*} > c\Big) \le c^{-1} E\Big\{\frac{1 + (\theta \cdot S)_T}{W_T^*}\Big\} \le c^{-1}.$$

This clearly implies the boundedness of $\mathcal{K}(1)$ in probability and hence S satisfies NUPBR. This ends the proof of the theorem. $\square$

Lemma 4.4. *Let X be any RCLL semimartingale, and $\widetilde{\pi} \in L(X)$ such that $\mathcal{E}(\widetilde{\pi} \cdot X) > 0$. Then, the following are equivalent:*
(i) For any $\pi \in L(X)$ such that $\mathcal{E}(\pi \cdot X) \ge 0$, and any stopping time, τ, we have

$$E\Big[\frac{\mathcal{E}(\pi \cdot X)_\tau}{\mathcal{E}(\widetilde{\pi} \cdot X)_\tau}\Big] \le 1. \tag{4.38}$$

(ii) For any $\pi \in L(X)$ such that $\mathcal{E}(\pi \cdot X) \ge 0$, the ratio $\mathcal{E}(\pi \cdot X)/\mathcal{E}(\widetilde{\pi} \cdot X)$ is a supermartingale.

Proof. It is clear that $(ii) \Longrightarrow (i)$ is obvious and its proof will be omitted. Suppose that assertion (i) holds, and consider $\pi \in L(X)$ such that $\mathcal{E}(\pi \cdot X) \geq 0$. Then, for any τ and σ two stopping times such that $\tau \leq \sigma$ $P-a.s.$ and $A \in \mathcal{F}_\tau$, we put

$$\overline{\pi} := \widetilde{\pi} I_{]\!]0,\tau_A]\!]} + \pi I_{]\!]\tau_A,+\infty[\![}, \quad \tau_A := \begin{cases} \tau & \text{on } A \\ +\infty & \text{on } A^c \end{cases}$$

Then, we easily calculate

$$\frac{\mathcal{E}(\overline{\pi} \cdot X)_\sigma}{\mathcal{E}(\widetilde{\pi} \cdot X)_\sigma} = \frac{\mathcal{E}(\widetilde{\pi} \cdot X)_\tau}{\mathcal{E}(\widetilde{\pi} \cdot X)_\sigma} \frac{\mathcal{E}(\pi \cdot X)_\sigma}{\mathcal{E}(\pi \cdot X)_\tau} I_A + I_{A^c}.$$

Therefore, a direct application of (4.38) for $\overline{\pi}$ and σ, we obtain

$$E\Big\{\frac{\mathcal{E}(\widetilde{\pi} \cdot X)_\tau}{\mathcal{E}(\widetilde{\pi} \cdot X)_\sigma} \frac{\mathcal{E}(\pi \cdot X)_\sigma}{\mathcal{E}(\pi \cdot X)_\tau} I_A\Big\} \leq P(A),$$

for any $A \in \mathcal{F}_\tau$. Hence, the supermartingale property for $\mathcal{E}(\pi \cdot X)\Big(\mathcal{E}(\widetilde{\pi} \cdot X)\Big)^{-1}$ follows immediately, and the proof of the lemma is achieved. ☐

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